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On the Extended Affine Root System

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§1. Intrtoduction.

Coxeter transformation plays an important role in the finite reflexion group theory. A new class of reflexion groups (which are not of finite ones) was defined and studied in [S -3],[S-4]. These groups also have the Coxeter transformations.

In this note, we define the extended affine root system and state the Coxeter transformation for the extended affine root system ([S-3]) . Also we define the flat invariants as an application of the Coxeter transformation theory ([S-4]) . Moreover we define the automorphism group of the extended affine root system and study its action on the flat invariants ([Sa]). The action of the automorphism group gives the modular property for the flat invariants (some theta functions).

The authors gave three lectures in the meeting. This note based on the lectures is written by the second author.

§2. Review of notations of extended affine root system.

We prepare some notations from Saito [S-3],[S-4]. For details, one is refered to [S-3],[S-4].

(2.1) Definition of extended affine root system.

Let F be a real vector space of rank $l+2$ with a positive semi-definite symmetric bilinear form $I : F \times F \to \mathbf{R}$, whose radical: $rad(I) := \{x \in F : I(x,y) = 0 \text{ for } {}^{\forall}y \in F\}$, is a vector space of rank 2. For a non-isotropic element $\alpha \in F(i.e. I(\alpha,\alpha) \neq 0)$, put $\alpha^{\vee} := 2\alpha/I(\alpha,\alpha) \in F$. The reflection w_α with respect to α is an element of $O(F,I) := \{g \in GL(F) : I(x,y) = I(g(x),g(y))\}$ given by ,

$$w_\alpha(u) := u - I(u,\alpha^{\vee})\alpha \ ({}^{\forall}u \in F).$$

Then $\alpha^{\vee\vee} = \alpha$ and $w_\alpha^2 = identity$.

Definition 2.1.

1. *A set R of non-isotropic elements of F is an extended affine root system belonging to(F,I), if it satisfies the axioms 1)-4:*
   1) *The additive group generated by R in F, denoted by $Q(R)$, is a full sub-lattice of F. I.e.,the embedding $Q(R) \subset F$ induces the isomorphism: $Q(R) \otimes_{\mathbf{Z}} \mathbf{R} \simeq F$.*
   2) *$I(\alpha, \beta^\vee) \in \mathbf{Z}$ for $^\forall\alpha, \beta \in R$.*
   3) *$w_\alpha(R) = R$ for $^\forall\alpha \in R$.*
   4) *If $R = R_1 \cup R_2$ with $R_1 \perp R_2$, then either R_1 or R_2 is void.*
2. *A marking G for the extended affine root system is a rank 1 subspace of $rad(I)$ such that $G \cap Q(R) \simeq \mathbf{Z}$.*

The pair (R, G) will be called a marked extended affine root system. Two marked extended affine root systems are isomorphic, if there exists a linear isomorphism of the ambient vector spaces, inducing the bijection of the sets of roots and the markings. A generator of $G \cap Q(R) \simeq \mathbf{Z}$, which is unique up to a sign, is denoted by a.

$$G \cap Q(R) = \mathbf{Z}a \text{ and } G = \mathbf{R}a.$$

remark 1. *For a root system R belonging to (F, I), there exists a real number $c > 0$ such that the bilinear form cI defines an even lattice structure on $Q(R)$ (i.e. $cI(x, x) \in 2\mathbf{Z}$ for $x \in Q(R)$). The smallest such c is denoted by $(I_R : I)$ and the bilinear form $(I_R : I)I$ is denoted by I_R.*

remark 2. *$w_\alpha(\alpha) = -\alpha$. Thus the multiplication of -1 is an automorphism of the extended affine root system.*

remark 3. *If $u \in rad(I)$, then $w_\alpha(u) = u - I(u, \alpha^\vee)\alpha = u$. Thus the Weyl group is identity on the $rad(I)$.*

remark 4. *If R is a root system belonging to (F, I), then $R^\vee := \{\alpha^\vee : \alpha \in R\}$ is also a root system belonging to (F, I).*

remark 5. *For a root system R belonging to (F, I), there exists a positive integer $t(R)$ such that*

$$I_R \otimes I_{R^\vee} = t(R) I \otimes I.$$

$t(R)$ is called the tier number of R.

(2.2) The basis $\alpha_0, \ldots, \alpha_l$ for (R, G).

The image of R by the projection $F \to F/rad(I)$ (resp. $F \to F/G$) is a finite (resp.affine) root system,which we shall denote by R_f (resp. R_a). In this paper, we assume that the affine root system R_a is reduced. (I.e. $\alpha = c\beta$ for $\alpha, \beta \in R_a$ and $c \in \mathbf{R}$ implies $c \in \{\pm 1\}$.)

Once and for all in this paper, we fix $l + 1$ elements,

$$\alpha_0, \ldots, \alpha_l \in R$$

such that their images in R_a form a basis for R_a [Mac]. We shall call them a basis for (R, G). Such basis is unique up to isomorphisms of (R, G). There exists positive integers $n_0, \ldots, n_l$ such that the sum:

$$b := \sum_{i=0}^{l} n_i \alpha_i \tag{2.2.1}$$

belongs to $rad(I)$. By a permutation of this basis,we may assume [Mac],

$$n_0 = 1. \tag{2.2.2}$$

Then the images of $\alpha_1, \ldots, \alpha_l$ in R_f form a positive basis for R_f and the image of $-\alpha_0$ in R_f is the highest root with respect to the basis. Put,

$$L := \bigoplus_{i=0}^{l} \mathbf{R}\alpha_i \tag{2.2.3}$$

on which I is positive definite and $R \cap L$ is a finite root system with the positive basis $\alpha_1, \ldots, \alpha_l$.

We have a direct sum decomposition of the vector space:

$$F = L \oplus rad(I), \tag{2.2.4}$$

and the lattice;

$$Q(R) = \bigoplus_{i=0}^{l} \mathbf{Z}\alpha_i \oplus \mathbf{Z}a = \bigoplus_{i=1}^{l} \mathbf{Z}\alpha_i \oplus \mathbf{Z}a \oplus \mathbf{Z}b, \tag{2.2.5}$$

$$Q(R) \cap rad(I) = \mathbf{Z}a \oplus \mathbf{Z}b, \tag{2.2.6}$$

$$Q(R) \cap L = \bigoplus_{i=1}^{l} \mathbf{Z}\alpha_i. \tag{2.2.7}$$

remark. *The choice of the basis $\alpha_0, \ldots, \alpha_l$ is done for the sake of explicit calculation, but it does not affect the result of the present paper. A change of the basis $\alpha_0, \ldots, \alpha_l$ induces a change (a, b) to $(a, b + ma)$ for some $m \in \mathbf{Z}$.*

(2.3) The Weyl group W_R.

The Weyl group W_R for R is defined as the group generated by the reflexion w_α for $^\forall\alpha \in R$. The projection $p : F \to F/rad(I)$ induces a homomorphism $p_* : W_R \to W_{R_f}$. One gets a short exact sequence:

$$0 \longrightarrow H_R \xrightarrow{E} W_R \xrightarrow{p_*} W_{R_f} \longrightarrow 1. \tag{2.3.1}$$

Here

$$H_R := (rad(I) \otimes_R F/rad(I)) \cap E^{-1}(W_R) \tag{2.3.2}$$

is a finite index subgroup in the lattice $(\mathbf{Z}a \oplus \mathbf{Z}b) \otimes_{\mathbf{Z}} (\oplus_{i=1}^{l} \mathbf{Z}\alpha_i^\vee)$.

The map E called the Eichler-Siegel transformation, is a semi-group homomorphism defined as follows([S-3]).

$$E : F \otimes_{\mathbf{R}} F/rad(I) \to End(F) \tag{2.3.3}$$

$$E(\sum_i \xi_i \otimes \eta_i)(u) := u - \sum_i \xi_i I(\eta_i, u) \text{ for } u \in F. \tag{2.3.4}$$

Here a semi-group structure $\circ$ on $F \otimes_{\mathbf{R}} F/rad(I)$ is defined by,

$$(2.3.5) \quad (\sum_i u_i \otimes v_i) \circ (\sum_j w_j \otimes x_j) := \sum_i u_i \otimes v_i + \sum_j w_j \otimes x_j - \sum_{i,j} I(v_i, w_j) u_i \otimes x_j.$$

The semi-group structure $\circ$ coincides with the natural addition of vectors on the subspace $rad(I) \otimes (F/rad(I))$ and hence on H_R.

(2.4) The Dynkin graph.

For a marked extended affine root system (R, G), we associate a diagram $\Gamma_{(R,G)}$, called the Dynkin graph for (R, G), in which all data on (R, G) are coded. The graph is constructed in the following steps 1)-4).

1) Let Γ be the graph for the affine root system $(R_a, F/G)$,*i.e.*

   a) The set of the vertices $|\Gamma|$ is $\{\alpha_0, \cdots, \alpha_l\}$.

   b) Edges of Γ is given according to a convention in 4) b).

2) The exponent for each vertex $\alpha_i \in |\Gamma|$ is defined by

$$(2.4.1) \qquad m_i := \frac{I_R(\alpha_i, \alpha_i)}{2k(\alpha_i)} n_i,$$

   where $k(\alpha) := inf\{n \in \mathbf{N} : \alpha + na \in R\}$.

3) Put

$$m_{max} := max\{m_0, \cdots, m_l\},$$

$$|\Gamma_m| := \{\alpha_i \in |\Gamma| : m_i = m_{max}\},$$

$$|\Gamma_m^*| := \{\alpha_i + k(\alpha_i)a : \alpha_i \in |\Gamma_m|\}.$$

4) The graph $\Gamma_{R,G}$ is defined as the graph for $|\Gamma| \cup |\Gamma_m^*|$,*i.e.*

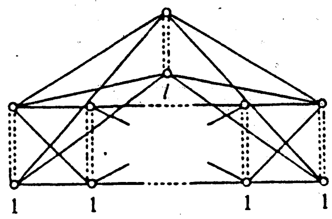
   a) The set of the vertices $|\Gamma_{R,G}| := |\Gamma| \cup |\Gamma_m^*|$.

   b) Two vertices $\alpha, \beta \in |\Gamma_{R,G}|$ are connected by the convention:

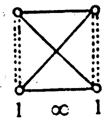
$\overset{\alpha}{\circ} \qquad \overset{\beta}{\circ}$	if $I(\alpha, \beta^\vee) = 0 (\Leftrightarrow I(\beta, \alpha^\vee) = 0)$,
$\circ\!\!-\!\!\!-\!\!\!-\!\!\circ$	if $I(\alpha, \beta^\vee) = I(\beta, \alpha^\vee) = -1$,
$\circ\!\!-\!\!\underset{t}{\leftarrow}\!\!-\!\!\circ$	if $I(\alpha, \beta^\vee) = -1, I(\beta, \alpha^\vee) = -t$,
$\circ\!\!\underset{\infty}{-\!\!\!-\!\!\!-}\!\!\circ$	if $I(\alpha, \beta^\vee) = I(\beta, \alpha^\vee) = -2$,
$\circ \cdots \circ$ (double dotted)	if $I(\alpha, \beta^\vee) = I(\beta, \alpha^\vee) = 2$.

Table 1. Dynkin Diagrams for Extended Affine Root Systems and Their Exponents

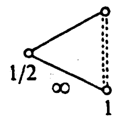
$A_l^{(1,1)}$



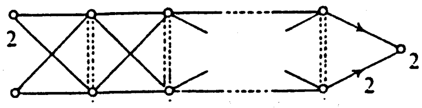
$(l \geq 2)$



$A_1^{(1,1)*}$

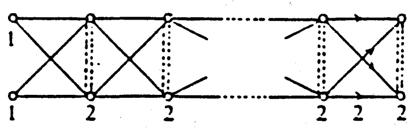


$B_l^{(1,1)}$



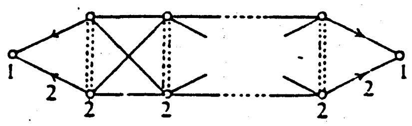
$(l \geq 3)$

$B_l^{(1,2)}$



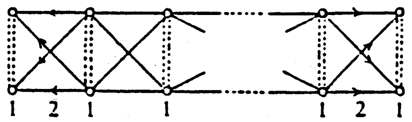
$(l \geq 3)$

$B_l^{(2,1)}$



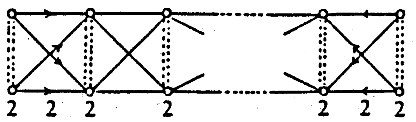
$(l \geq 2)$

$B_l^{(2,2)}$



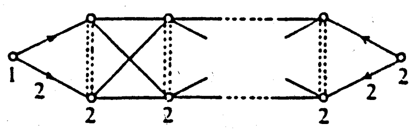
$(l \geq 2)$

$C_l^{(1,1)}$



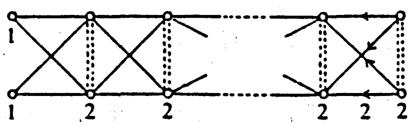
$(l \geq 2)$

$C_l^{(1,2)}$



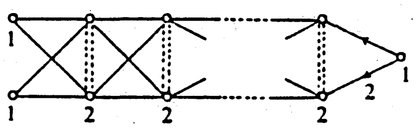
$(l \geq 2)$

$C_l^{(2,1)}$



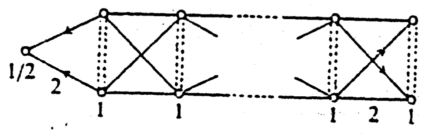
$(l \geq 3)$

$C_l^{(2,2)}$



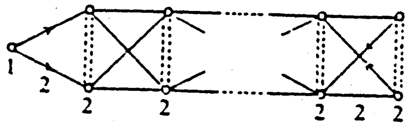
$(l \geq 3)$

$B_l^{(2,2)}$



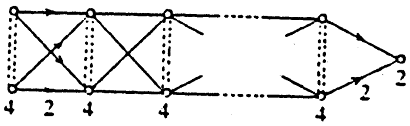
$(l \geq 2)$

$C_l^{(1,1)}$

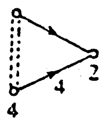


$(l \geq 2)$

$BC_l^{(2,1)}$

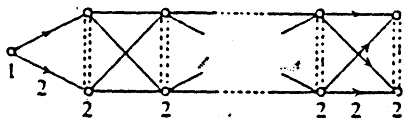


$(l \geq 2)$,

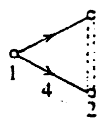


$(l = 1)$

$BC_l^{(2,1)}$

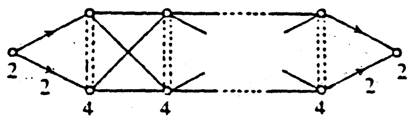


$(l \geq 2)$,



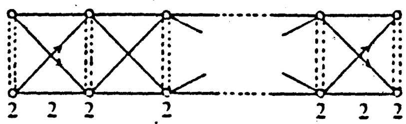
$(l = 1)$

$BC_l^{(2,2)}(1)$

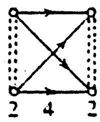


$(l \geq 2)$

$BC_l^{(2,2)}(2)$

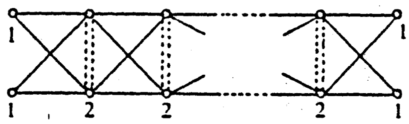


$(l \geq 2)$,



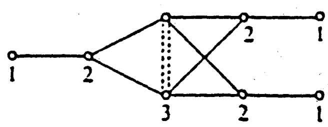
$(l = 1)$

$D_l^{(1,1)}$

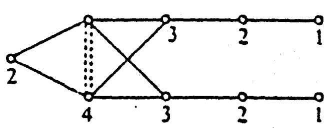


$(l \geq 4)$

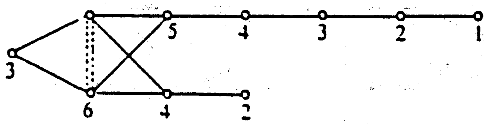
$E_6^{(1,1)}$



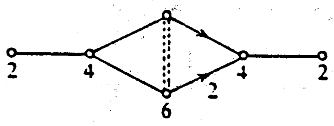
$E_7^{(1,1)}$



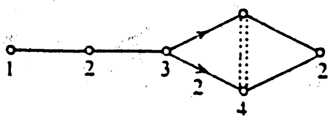
$E_8^{(1,1)}$



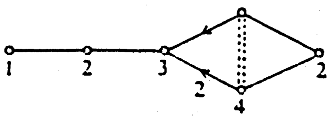
$F_4^{(1,1)}$



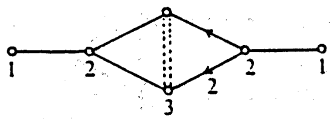
$F_4^{(1,2)}$



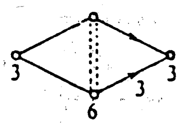
$F_4^{(2,1)}$



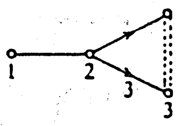
$F_4^{(2,2)}$



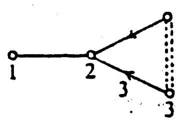
$G_2^{(1,1)}$



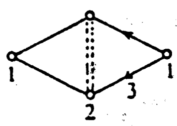
$G_2^{(1,3)}$



$G_2^{(1,1)}$



$G_2^{(3,3)}$



Definition. *For a marked extended affine root system (R, G), the codimension, denoted by $cod(R, G)$, is defined as follows.*

$$cod(R,G) := \#\{0 \le i \le l : m_i = m_{max}\} = \#|\Gamma_m|. \tag{2.4.2}$$

Note. *The exponents m_i's introduced in 2) are half integers, which might have a common factor. We have : The smallest common denominator for the rational numbers $m_i/m_{max}(i = 0, \cdots, l)$ is equal to $l_{max} + 1$([S-3]), where $l_{max} :=$ max { # of vertices in a connected component of $\Gamma \setminus \Gamma_m$}. Thus we sometimes normalize the exponents as follows.*

$$\tilde{m}_i := m_i \frac{l_{max}+1}{m_{max}} \ (i = 0, \cdots, l). \tag{2.4.3}$$

(2.5) The Coxeter transformation for (R, G).

A Coxeter transformation $c \in W_R$ is, by definition [S-3], a product of reflexions w_α for $\alpha \in |\Gamma_{R,G}|$ with a restriction on the order of the product that w_{α^*} comes next to w_α for $\alpha \in |\Gamma_m|$. The following Lemma's A, B and C are basic results for the Coxeter transformation, which will be used essentially in this note.

Lemma A ([S-3](9.7)). *A Coxeter transformation c is semi-simple of finite order $=$ $l_{max} + 1$. The set of eigenvalues of c is given by:*

$$1 = exp(0) \text{ and } exp(2\pi\sqrt{-1}m_i/m_{max})(i = 1, \cdots, l).$$

Particularly, the multiplicity of eigenvalue 1= $1 + cod(R, G)$.

Lemma B ([S-3](10.1)). *Let c be a Coxeter transformation for (R, G). Then*

$$R \cap Image(c - id_F) = \phi.$$

(2.6) The hyperbolic extension $(\tilde{F}, \tilde{I})$.

There exists uniquely (up to a linear isomorphism) a real vector space $\tilde{F}$ of rank $l + 3$ with

1) an inclusion map $F \subset \tilde{F}$ as a real vector space,

2) a symmetric form $\tilde{I} : \tilde{F} \times \tilde{F} \to \mathbf{R}$ such that $\tilde{I}|_F = I$ and $rad(I) = \mathbf{R}a$.

The pair $(\tilde{F}, \tilde{I})$ will be called a hyperbolic extension for (F, I).

Denote by $\tilde{w}_\alpha$ the reflexion for $\alpha \in R$ as an element of $GL(\tilde{F})$ and by $\tilde{W}_R$ the subgroup of $O(\tilde{F}, \tilde{I})$(where, $O(\tilde{F}, \tilde{I}) := \{g \in GL(\tilde{F}) | \tilde{I}(x, y) = \tilde{I}(gx, gy) \forall x, y \in \tilde{F}\}$.) generated by them. The restriction $\tilde{w}_\alpha|_F$ is w_α. Thus we have a surjection $\tilde{W}_R \to W_R$ and then a short exact sequence:

(2.6.1)
$$0 \longrightarrow \tilde{K}_R \xrightarrow{\tilde{E}} \tilde{W}_R \longrightarrow W_R \longrightarrow 1$$

where $\tilde{K}_R$ is an infinite cyclic group generated by

(2.6.2)
$$k := (I : I_R) \frac{l_{max} + 1}{m_{max}} a \otimes b,$$

and $\tilde{E} : F \otimes F/G \to End(\tilde{F})$ is the Eichler-Siegel transformation,

(2.6.3)
$$\tilde{E}(\sum_i \xi_i \otimes \eta_i)(u) := u - \sum_i \xi_i \tilde{I}(\eta_i, u) \text{ for } u \in \tilde{F}.$$

$\tilde{H}_R$ is a subgroup of $\tilde{W}_R$ defined as a kernel of the composite map:

$$\tilde{W}_R \longrightarrow W_R \xrightarrow{p_*} W_{R_f}.$$

We have the following diagram.

(2.6.4)
$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{K}_R & \longrightarrow & \tilde{H}_R & \longrightarrow & H_R & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow E & & \\
0 & \longrightarrow & \tilde{K}_R & \xrightarrow{\tilde{E}} & \tilde{W}_R & \longrightarrow & W_R & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow p_* & & \\
 & & & & W_{R_f} & = & W_{R_f} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

(2.7) The hyperbolic Coxeter transformation.

A hyperbolic Coxeter transformation $\tilde{c} \in \tilde{W}_R$ is a product of reflexions $\tilde{w}_\alpha$ for $\alpha \in |\Gamma_{R,G}|$ in the same ordering as for the Coxeter transformation c ([S-3](11.2)).

Lemma C ([S-3](11.3)).

1) The power $\tilde{c}^{(l_{max}+1)}$ of the hyperbolic Coxeter transformation $\tilde{c}$ is a generator of $\tilde{K}$.

2) $\tilde{K}$ is generated by $(I_R : I)\frac{l_{max}+1}{m_{max}} a \otimes b$.

This is equivalent to:

Lemma C' **([S-3](11.4.1)).** *There exists a projection map $p : \tilde{F} \to F$ such that for $^{\forall}\tilde{\lambda} \in \tilde{F}$,*

$$\tilde{c}(\tilde{\lambda}) = \tilde{\lambda} + (c - id_F)p(\tilde{\lambda}) + \tilde{I}(b, \tilde{\lambda})\frac{(I_R : I)}{m_{max}} a. \tag{2.7.1}$$

(2.8) A family of polarized Abelian variety over H.

Let (R, G) be a marked extended affine root system and let $(\tilde{F}, \tilde{I})$ be its hyperbolic extension. We define complex affine half spaces as follows.

$$\tilde{\mathbf{E}} := \{x \in Hom_{\mathbf{R}}(\tilde{F}, \mathbf{C}) : a(x) = 1 \text{ and } Im(b(x)) > 0\}, \tag{2.8.1}$$

$$\mathbf{E} := \{x \in Hom_{\mathbf{R}}(F, \mathbf{C}) : a(x) = 1 \text{ and } Im(b(x)) > 0\}, \tag{2.8.2}$$

$$\mathbf{H} := \{x \in Hom_{\mathbf{R}}(rad(I), \mathbf{C}) : a(x) = 1 \text{ and } Im(b(x)) > 0\}, \tag{2.8.3}$$

where $dim_{\mathbf{C}}\tilde{\mathbf{E}} = l + 2$, $dim_{\mathbf{C}}\mathbf{E} = l + 1$, and $dim_{\mathbf{C}}\mathbf{H} = 1$. A change of the basis $\alpha_0, \ldots, \alpha_l$ does not affect the definition of the spaces $\tilde{\mathbf{E}}, \mathbf{E}, \mathbf{H}$. The inclusion maps : $\tilde{F} \supset F \supset rad(I)$ induces the projections:

$$\tilde{\mathbf{E}} \xrightarrow{\tilde{\pi}} \mathbf{E} \xrightarrow{\pi} \mathbf{H}. \tag{2.8.4}$$

By the projection, $\tilde{\mathbf{E}}$ and $\mathbf{E}$ are regarded as a total space of a family of complex affine spaces $\tilde{\mathbf{E}}_\tau := (\pi \circ \tilde{\pi})^{-1}(\tau)$ and $\mathbf{E}_\tau := \pi^{-1}(\tau)$ of dimension $l + 1$ and l parametrized by $\tau \in \mathbf{H}$, and $\tilde{\mathbf{E}}$ has an affine bundle structure on $\mathbf{E}$. The action of the groups W_R and $\tilde{W}_R$ on F and $\tilde{F}$ fixes the $rad(I)$ pointwisely. Hence the contragredient actions of W_R and $\tilde{W}_R$ induces actions on $\mathbf{E}$ and $\tilde{\mathbf{E}}$ respectively. They are equivarient with the projections $\tilde{\pi}$ and π (2.8.4).

Lemma 2.1(Saito [S-4]).

1. *The actions of W_R (resp. $\tilde{W}_R$) on $\mathbf{E}$ (resp. $\tilde{\mathbf{E}}$) are properly discontinuous.*
2. *Put $X := \mathbf{E}/H_R$ and denote by π/H_R the map induced from π:*

$$(2.8.5) \qquad \pi/H_R : X \to \mathbf{H}.$$

The fiber $X_\tau := (\pi/H_R)^{-1}(\tau)$ over $\tau \in \mathbf{H}$ is isogeneous to an l-times product of elliptic curves of the same modulous τ.

3. *The action of $\tilde{H}_R$ on $\tilde{\mathbf{E}}$ is fixed point free. Put $L^* := \tilde{\mathbf{E}}/\tilde{H}_R$. The map $\tilde{\pi}/\tilde{H}_R$ induced from $\tilde{\pi}$:*

$$(2.8.6) \qquad \tilde{\pi}/\tilde{H}_R : L^* \to X,$$

defines a principal $\mathbf{C}^*$-bundle over X. Let L be the associated complex line bundle over X , which is, as a set, a union

$$(2.8.7) \qquad L = L^* \cup X.$$

The finite Weyl group W_{R_f} is acting on L and X equivariantly.

4. *The Chern class $c(L|_{X_\tau})$ of the line bundle over $X_\tau := (\pi/H_R)^{-1}(\tau)$ for $\tau \in \mathbf{H}$ is given by,*

$$(2.8.8) \qquad c(L|_{X_\tau}) = Im(H) \in \bigwedge^2 Hom_{\mathbf{Z}}(H_R, \mathbf{C}) \simeq H^2(X_\tau, \mathbf{C}),$$

where H is an Hermitian form on $V_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{R}} (F/rad(I))^*$ given by

$$(2.8.9) \qquad H(z,w) := -\frac{m_{max}}{t(R)(l_{max}+1)Im(\tau)} I_{R^\vee}(z, \bar{w}).$$

remark. *Since the line bundle L^{-1} is ample relative to $\mathbf{H}$, one may blow down the zero section $X \subset L$ of L to $\mathbf{H}$ (i.e.X_τ is blow down to a point for $\tau \in \mathbf{H}$). The blow down space, denoted as*

$$(2.8.10) \qquad \mathbf{L}(\simeq L^* \cup \mathbf{H}),$$

is a family of affine algebraic variety of dimension $l+1$ with an isolated singularity parametrized by the space $\mathbf{H}$.

(2.9) Chevalley type theorem.

In this subsection, we recall a Chevalley type theorem (Theorem 2.2), studied by Looijenga[L-2], Schwarzman & Bernstein [B-S1],[B-S2], Kac & Peterson [K-P] and others.

Let us fix a base $\tilde{\lambda} \in \tilde{F} \setminus F$ normalized as

(2.9.1) $$\tilde{I}(\tilde{\lambda}, b) = 1,$$

(2.9.2) $$\tilde{I}(\tilde{\lambda}, \alpha_i) = 0.(0 \le i \le l)$$

Consider this $\tilde{\lambda}$ as a complex coordinate for $\tilde{\mathbf{E}}$, we obtain

(2.9.3) $$(\tilde{\lambda}, \tilde{\pi}) : \tilde{\mathbf{E}} \simeq \mathbf{C} \times \mathbf{E}.$$

The generater k of $\tilde{K_R}$ acts on $\tilde{\lambda}$ by

(2.9.4) $$\tilde{\mathbf{E}}(k)(\tilde{\lambda}) = \tilde{\lambda} - (I_R : I)\frac{l_{max}+1}{m_{max}}a.$$

Hence the complex function λ on $\tilde{\mathbf{E}}$ defined by

(2.9.5) $$\lambda := exp\left(2\pi\sqrt{-1}\frac{m_{max}}{(I_R : I)(l_{max}+1)}\tilde{\lambda}\right),$$

is $\tilde{K_R}$ invariant, giving a fiber coordinate for the $\mathbf{C}^*$ bundle:

(2.9.6) $$(\lambda, \tilde{\pi}) : \tilde{\mathbf{E}}/\tilde{K_R} \simeq \mathbf{C}^* \times \mathbf{E}.$$

For a non negative integer k, let

(2.9.7) $$S_k := \Gamma(X, \mathcal{O}(L^{-\otimes k}))$$

be the module of holomorphic sections of the $-k$ th power of the line bundle L over X defined in (2.8) Lemma 2.1. For an element $\Theta \in S_k$, put

(2.9.8) $$\tilde{\Theta} := (\lambda)^k \Theta.$$

Then $\tilde{\Theta}$ is a $\tilde{H}_R$-invariant holomorphic function on $\tilde{\mathbf{E}}$. The group $W_{R_f} \simeq \tilde{W}_R/\tilde{H}_R$ acts on L and X equivariantly. Therefore W_{R_f} acts on the space of sections S_k, ($k = 0, 1, \ldots$). Put

(2.9.9) $\quad S_k^W :=$ the set of W_{R_f} invariant elements of S_k.

(2.9.10) $\quad S_k^{-W} :=$ the set of W_{R_f} anti-invariant elements of S_k.

(2.9.11) $$S^W := \bigoplus_{k=0}^{\infty} S_k^W.$$

(2.9.12) $$S^{-W} := \bigoplus_{k=0}^{\infty} S_k^{-W}.$$

Naturally S^W is a $\Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}})$-graded algebra, and the grading is defined by k. We prepare one more concept: the Jacobian $J(\Theta_1, \ldots, \Theta_{l+2})$ for a system of sections $\Theta_i \in S_{k_i}(i = 1, \ldots, l+2)$ as an element of $S_k(k = \sum_{i=1}^{l+2} k_i)$ given by the following relation.

(2.9.13) $$d\tilde{\Theta}_1 \wedge \ldots d\tilde{\Theta}_{l+2} = \tilde{J}(\Theta_1, \ldots, \Theta_{l+2})(d\tau \wedge d\alpha_1 \wedge \ldots, d\alpha_l \wedge d\tilde{\lambda}).$$

The Jacobian is well defined, since $\omega := d\tau \wedge d\alpha_1 \wedge \ldots, d\alpha_l \wedge d\tilde{\lambda}$ is $\tilde{H}_R$-invariant. Moreover, since the form ω is $\tilde{W}_R$ anti- invariant and $\Theta_i \in S_{k_i}^W(i = 1, \ldots, l+2)$, thus $J(\Theta_1, \ldots, \Theta_{l+2})$ $\in S_k^{-W}(k = \sum_{i=1}^{l+2} k_i)$, (where $\tilde{J} = \lambda^k J$).

Theorem 2.2 ([B-S1][B-S2][L-2][K-P]).

1. S^W *is a polynomial algebra over* $\Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}})$, *freely generated by* $l+1$ *homogeneous elements* $\Theta_0, \ldots \Theta_l$ *of degree* $\tilde{m}_i := m_i \frac{l_{max}+1}{m_{max}} (i = 0, \ldots, l)$, *where* $m_i (i = 0, \ldots, l)$ *is the set of exponents for the root system* (R, G).
2. S^{-W} *is a free* S^W*-module of rank 1 generated by* $\Theta_A := J(\tau, \Theta_0, \ldots, \Theta_{l+l})$ *homogeneous of degree* $\frac{(l+1+cod(R,G))(l_{max}+1)}{2}$.
3. *The zero-loci of* Θ_A *on* $\tilde{\mathbf{E}}$ *is equal to the union* ${}_{\alpha \in R} \cup H_\alpha$ *of the complex hyperplanes* H_α *defined by the discriminant for* (R, G).

remark. *As an analogue to finite reflexion group case, we ask to clarify the relationship among the following three polynomials:*

1) *The Möbius function for the lattice defined by the system of hyperplanes* $\mathbf{H}_\alpha (\alpha \in R)$ *in* $\mathbf{L}$.

2) *The Poincaré polynomial for the topological space* $\mathbf{L} \setminus \bigcup_{\alpha \in R} \mathbf{H}_\alpha$.

3) $P(T) := \prod_{i=1}^{l} (1 + \tilde{m}_i T)$.

(For finite reflexion group case, these polynomials coincide. (see Terao[T], Orlik-Solomon [O-S1],[O-S2].)

(2.10) The C-metrics $\tilde{I}_W, \tilde{I}_W^*$.

Let us denote by $\mathcal{O}_{\tilde{\mathbf{E}}}, \Omega^1_{\tilde{\mathbf{E}}}$ and $Der_{\tilde{\mathbf{E}}}$ the sheaf of germs of holomorphic functions, 1-forms and vector fields on $\tilde{\mathbf{E}}$ respectively. Since $\tilde{\mathbf{E}}$ is a complex affine space, the tangent and co-tangent spaces of $\tilde{\mathbf{E}}$ is naturally given by:

(2.10.1) $$T_x(\tilde{\mathbf{E}}) \simeq \mathbf{C} \otimes_{\mathbf{R}} (\tilde{F}/G)^*,$$

(2.10.2) $$T_x^*(\tilde{\mathbf{E}}) \simeq \mathbf{C} \otimes_{\mathbf{R}} (\tilde{F}/G).$$

Thus we have the canonical isomorphisms:

(2.10.3) $$\Omega^1_{\tilde{\mathbf{E}}} \simeq \mathcal{O}_{\tilde{\mathbf{E}}} \otimes_{\mathbf{R}} (\tilde{F}/G) \text{ and } Der_{\tilde{\mathbf{E}}} \simeq \mathcal{O}_{\tilde{\mathbf{E}}} \otimes_{\mathbf{R}} (\tilde{F}/G)^*.$$

The vector space $(\tilde{F}/G)$ carries a nondegenerate symmetric bilinear form induced from $\tilde{I}$. By extending $\tilde{I}$ to $\Omega^1_{\tilde{\mathbf{E}}}$ by $\mathcal{O}_{\tilde{\mathbf{E}}}$-bilinearly, we obtain a form:

(2.10.4) $$\begin{array}{rcl} \tilde{I}_{\tilde{\mathbf{E}}} : \Omega^1_{\tilde{\mathbf{E}}} \times \Omega^1_{\tilde{\mathbf{E}}} & \to & \mathcal{O}_{\tilde{\mathbf{E}}} \\ \omega_1 \times \omega_2 & \mapsto & \sum_{i,j=1}^{l+2} \frac{\omega_1}{\partial X_i} \frac{\omega_2}{\partial X_j} \tilde{I}(X_i, X_j), \end{array}$$

where $X_i (i = 1, \cdots, l+2)$ are basis of $\tilde{F}/G$ and $\omega = \sum_i \frac{\omega}{\partial X_i} dX_i$. Put,

(2.10.5) $$Der_{S^W} := \text{ the module of } \mathbf{C}\text{-derivations of the algebra } S^W,$$

(2.10.6) $$\Omega^1_{S^W} := \text{ the module of 1-forms for the algebra } S^W.$$

They are dual S^W-free modules by the natural pairing: $<,>$ with the dual basis:

(2.10.7) $$Der_{S^W} = S^W \frac{\partial}{\partial \tau} \oplus \bigoplus_{i=0}^{l} S^W \frac{\partial}{\partial \Theta_i},$$

(2.10.8) $$\Omega^1_{S^W} = S^W d\tau \oplus \bigoplus_{i=0}^{l} S^W d\Theta_i$$

using a generator system Θ_i's of Theorem 2.2. Der_{S^W} and $\Omega^1_{S^W}$ have the graded S^W-module structure in a natural way. There is a natural lifting map:

$$\begin{array}{ccc} \Omega^1_{S^W} & \to & \Omega^1_{\tilde{\mathbf{E}}}, \\ d\tilde{\Theta} & \mapsto & \sum_i \frac{\partial \tilde{\Theta}}{\partial X_i} dX_i, \end{array} \tag{2.10.9}$$

so that the form $\tilde{I}_{\tilde{\mathbf{E}}}$ induces a S^W-bilinear form:

$$\tilde{I}_W : \Omega^1_{S^W} \times \Omega^1_{S^W} \to S^W. \tag{2.10.10}$$

(The values of $\tilde{I}_W$ lie in S^W, since the form $\tilde{I}_{\tilde{\mathbf{E}}}$ is $\tilde{W}_R$ invariant.) Let us denote by $\tilde{I}_W$ the S^W-bilinear form on the module Der_{S^W} dual to the form $\tilde{I}_W$. We use the next lemma in section 4.

Lemma([S-4]).

$$\tilde{I}_W(d\tau, d\tilde{\Theta}) = \kappa^{-1}\Theta. \tag{2.10.11}$$

where $\Theta \in S^W_{\tilde{m}_l}$ *and* $\kappa := \frac{(I_R:I)}{2\pi\sqrt{-1}m_{max}}$.

Proof is easy. Thus we omit it.

§3.The automorphism group of the extended affine root system.

In this Section, we define the automorphism group $Aut^+(R)$ of R, and its central extension $\widetilde{Aut^+}(R)$, which act on $\mathbf{E}$ and $\tilde{\mathbf{E}}$ respectively. Also we show that $\widetilde{Aut^+}(R)$ contains $\tilde{W}_R$ as a normal subgroup.

(3.1) Definition of $Aut^+(R)$.

In this subsection, we introduce the automorphism group of R.

Definition 3.1. *For the extended affine root system $R \subset F$, put*

$$Aut(R) := \{g \in GL(F) | \ g \text{ induces a bijection of } R\}.$$

Proposition 3.2. *The extended affine Weyl group W_R is a normal subgroup of $Aut(R)$, and $Aut(R)$ is a subgroup of the orthogonal group $O(F, I)$.*

Proof. The latter part follows from Saito[S-3]. The first part follows from the formula: $g w_\alpha g^{-1} = w_{g\alpha}$ for $\alpha \in R$, $g \in Aut(R)$. *Q.E.D.*

The element g of the orthogonal group $O(F, I)$ induces the linear transformation of $rad(I)$ by restriction. We denote this restriction map by ρ.

$$\begin{array}{rccc} (3.1.1) \qquad & \rho : O(F, I) & \rightarrow & GL(rad(I)). \\ & g & \mapsto & g|_{rad(I)} \end{array}$$

Definition 3.3. $\Gamma := \rho(Aut(R))$.

Since each $\gamma \in \Gamma$ induces an isomorphism of $Q(R) \cap rad(I)$($\mathbf{Z}$ - free module of rank 2), determinant of γ equals ± 1. We shall consider only the elements whose determinant equals 1

Definition 3.4. *We define the following groups:*

$$\begin{aligned} SL(rad(I)) &:= \{g \in GL(rad(I)) | det g = 1\}, \\ O^+(F, I) &:= \rho^{-1}(SL(rad(I))), \\ O(F, rad(I)) &:= \rho^{-1}(1), \\ \Gamma^+ &:= \Gamma \cap SL(rad(I)), \\ Aut^+(R) &:= Aut(R) \cap O^+(F, I), \\ Aut(R, rad(I)) &:= Aut(R) \cap O(F, rad(I)). \end{aligned}$$

The relation between the Weyl group and these groups is as follows.

$$(3.1.2) \quad \begin{array}{ccccccccc} & & 1 & & 1 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & W_R & = & W_R & & & & \\ & & \downarrow & & \downarrow & & & & \\ 1 & \longrightarrow & Aut(R, rad(I)) & \longrightarrow & Aut^+(R) & \xrightarrow{\rho} & \Gamma^+ & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & Aut(R, rad(I))/W_R & \longrightarrow & Aut^+(R)/W_R & \xrightarrow{\rho} & \Gamma^+ & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & & & \\ & & 1 & & 1 & & & & \end{array}$$

remark. *The projection map $p : F \to F/rad(I)$ induces the homomorphism*

$$\tilde{p} : Aut(R, rad(I)) \to Aut(R_f),$$

where $R_f = p(R)$. Thus we have the following diagram.

$$(3.1.3) \quad \begin{array}{ccccccccc} & & 1 & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & H_R & \longrightarrow & W_R & \longrightarrow & W_f & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & ker\, \tilde{p} & \longrightarrow & Aut(R, rad(I)) & \xrightarrow{\tilde{p}} & Aut(R_f) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & ker\, \tilde{p}/H_R & \longrightarrow & Aut(R, rad(I))/W_R & \longrightarrow & Aut(R_f)/W_f & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 1 & & 1 & & 1 & & \end{array}$$

The abelian subgroup H_R becomes a finite index subgroup of $(\mathbf{Z}a \oplus \mathbf{Z}b) \otimes_{\mathbf{Z}} (\oplus_{i=1}^{l} \mathbf{Z}\alpha_i^\vee)$(see (2.1.2)), and kerp can be considered as a sublattice of $(\mathbf{Z}a \oplus \mathbf{Z}b) \otimes_{\mathbf{Z}} P$ where P is a dual *lattice of* $\oplus_{i=1}^{l}\mathbf{Z}\alpha_i$ *with respect to* $\bar{I}$ *, and* $\bar{I}$ *is a bilinear form on* $F/rad(I)$ *induced from* I*). Hence* $ker\ p/H_R$ *is a finite group. Furthermore* $Aut(R_f)/W_f$ *is a finite group (which is isomorphic to the automorphism group of finite Dynkin diagram corresponding to the finite Weyl group), therefore* $Aut(R, rad(I))/W_R$ *is also a finite group.*

(3.2) Explicit description of Γ^+.

We give an explicit description of Γ^+ for each marked extended affine root system. Fixing one basis $a, b \in rad(I) \cap Q(R)$ $(a \in G \cap Q(R))$, we can represent Γ^+ as a subgroup of $SL(2, \mathbf{Z})$.

1)$\Gamma^+ = SL(2,\mathbf{Z})$ for the cases $X_l^{(t,t)}, X_l^{(t,t)*}, (t = 1, 2, 3)$.

2)$\Gamma^+ = \{\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2,\mathbf{Z}) | q \equiv 0(mod\ 2)\}$ for the cases $B_l^{(1,2)}, C_l^{(1,2)}, F_4^{(1,2)}$.

3)$\Gamma^+ = \{\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2,\mathbf{Z}) | q \equiv 0(mod\ 3)\}$ for the case $G_2^{(1,3)}$.

4)$\Gamma^+ = \{\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2,\mathbf{Z}) | r \equiv 0(mod\ 2)\}$ for the cases $B_l^{(2,1)}, C_l^{(2,1)}, F_4^{(2,1)}$, $BC_l^{(2,4)}, BC_l^{(2,2)}(2)$.

5)$\Gamma^+ = \{\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2,\mathbf{Z}) | r \equiv 0(mod\ 3)\}$ for the case $G_2^{(3,1)}$.

6)$\Gamma^+ = \{\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2,\mathbf{Z}) | p \equiv 1(mod\ 2)\}$ for the cases $BC_l^{(2,1)}, BC_l^{(2,2)}(1)$.

(3.3) The action of $Aut^+(R)$ on E.

In order to define the action of $Aut^+(R)$ on the space $\mathbf{E}$, we introduce the space F^*_{half} as follows:

(3.3.1) $$F^*_{half} := \{x \in Hom_{\mathbf{R}}(F, \mathbf{C}) | < a, x > \neq 0, < b, x > \neq 0, Im\frac{< b, x >}{< a, x >} > 0\}.$$

The space F^*_{half} has a $\mathbf{C}^*$action induced from the $\mathbf{C}^*$ action on the complex vector space $Hom_{\mathbf{R}}(F, \mathbf{C})$, defined by

$$(\alpha f)(x) := \alpha(f(x)) \text{ for } f \in Hom_{\mathbf{R}}(F, \mathbf{C}), x \in F, \alpha \in \mathbf{C}^*.$$

We consider the next diagram.

(3.3.2)
$$\begin{tikzcd} & F^*_{half} \arrow[dr] & \\ \mathbf{E} \arrow[ur] & \simeq & F^*_{half}/\mathbf{C}^* \end{tikzcd}$$

The composite map of the $\mathbf{C}^*$ quotient map and the natural inclusion $\mathbf{E} \hookrightarrow F^*_{half}$ becomes an isomorphism. $O^+(F,I)$ acts on F^*_{half} contragrediently, thus $O^+(F,I)$ also acts on $F^*_{half}/\mathbf{C}^*$. Using the above isomorphism (3.3.2) , we can define the action of $O^+(F,I)$ and its subgroup $Aut^+(R)$ on $\mathbf{E}$. (We call this action "the linear fractional transformation".)

remark. *In the element of $O^+(F,I)$, only ± 1 can be considered as a $\mathbf{C}^*$-action. Therefore, $O^+(F,I)/\{\pm 1\}$ acts on the space $\mathbf{E}$ faithfully. The subgroup $O(F, rad(I))$ does not contain $-id.$, hence $O(F, rad(I))$ acts on the space $\mathbf{E}$ faithfully. We have the following diagram.*

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \{\pm 1\} & = & \{\pm 1\} & & \\
 & & & & \downarrow & & \downarrow & & \\
(3.3.3) \quad 1 & \to & O(F, rad(I)) & \to & O^+(F,I) & \xrightarrow{\rho} & SL(rad(I)) & \to & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \to & O(F, rad(I)) & \to & O^+(F,I)/\{\pm 1\} & \xrightarrow{\rho} & SL(rad(I))/\{\pm 1\} & \to & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

$O^+(F,I)/\{\pm 1\}$ acts on $\mathbf{E}$ faithfully and transitively, and $SL(rad(I))/\{\pm 1\}$ also acts on $\mathbf{H}$ faithfully and transitively. Hence $\mathbf{E}$ and $\mathbf{H}$ have the structure of homogeneous space. These groups act on the bundle $\mathbf{E} \xrightarrow{\pi} \mathbf{H}$ equivariantly. Therefore we can regard $\mathbf{E} \xrightarrow{\pi} \mathbf{H}$ as the induced morphism by ρ.

(3.4) The central extension of $Aut^+(R)$.

In order to lift the action of $Aut^+(R)$ on $\mathbf{E}$ to $\tilde{\mathbf{E}}$, we need to define the central extension $\widetilde{Aut^+}(R)$ of $Aut^+(R)$. First, we define the central extension $\widetilde{O^\pm_{\mathbf{E}}}(F,I)$ of $O^+(F,I)/\{\pm 1\}$ (Definition 3.5) and also the central extension $\widetilde{O^+}(F,I)$ of $O^+(F,I)$ (Definition 3.6, Proposition 3.7). $\widetilde{Aut^+}(R)$ will be defined at Definition 3.8 as a subgroup of $\widetilde{O^+}(F,I)$.

We prepare an automorphic factor $(c\tau + d)^{-2}$ intrinsically. For any $f_0 \in O^+(F,I)/\{\pm 1\}$, f_0 acts on $\mathbf{E}$ as a bundle isomorphism of $\pi : \mathbf{E} \to \mathbf{H}$. We denote by $\bar{f}_0 (= \rho(f_0) \in SL(rad(I))/\{\pm 1\})$ the induced isomorphism of $\mathbf{H}$.

We can consider b, an element of the $\mathbf{Z}$ basis of $rad(I) \cap Q(R)$(introduced in (2.2.1)), as a coordinate function of $\mathbf{H}$. When we consider b as a coordinate function, we use the letter τ. Since b is unique up to adding $m \times a$ $(m \in \mathbf{Z})$, $d\tau$ and $\frac{\partial \bar{f}_0}{\partial \tau}(\tau)$ has an intrinsic meaning.

We recall that $\tilde{\mathbf{E}}$ has **the affine** complex line bundle structure over $\mathbf{E}$.

Now, we define **the central extention** of $O^+(F,I)/\{\pm 1\}$ as the subgroup of the holomorphic bundle iso**morphism of** $\tilde{\pi} : \tilde{\mathbf{E}} \to \mathbf{E}$ whose element satisfies the next conditions.

conditions. *For a* ***holomorphic*** *bundle map* $f : \tilde{\mathbf{E}} \to \tilde{\mathbf{E}}$, *there exists the unique* $f_0 \in O^+(F,I)/\{\pm 1\}$, *such* ***that***

$$1)\tilde{\pi} \circ f = f_0 \circ \tilde{\pi} \; on \; \tilde{\mathbf{E}} \tag{3.4.1}$$

$$2) f^* \tilde{I}^* = \frac{\partial \bar{f}_0}{\partial \tau}(\tau) \tilde{I}^*. \tag{3.4.2}$$

where $\tilde{I}^*$ *is a* $\mathbf{C}-$*metric on* $\tilde{E}$ *defined in (2.10.4).*

We denote the set of the above bundle maps by $\widetilde{O^+_{\tilde{\mathbf{E}}}}(F,I).i.e.$

Definition 3.5.

$\widetilde{O^+_{\tilde{\mathbf{E}}}}(F,I) := \{f : \tilde{\mathbf{E}} \to \tilde{\mathbf{E}}$ *holomorphic bundle isomorphism of* $\tilde{\pi} : \tilde{\mathbf{E}} \to \mathbf{E}$;

which satisfies the above conditions (3.4.1),(3.4.2).$\}$.

From the condition (3.4.1), we can define the homomorphism

$$\psi : \widetilde{O^+_{\tilde{\mathbf{E}}}}(F,I) \to O^+(F,I)/\{\pm 1\}. \tag{3.4.3}$$

Since $\frac{\partial \bar{f}_0}{\partial \tau}(\tau)$ satisfies the cocycle condition with respect to the group $SL(rad(I))$ $/\{\pm 1\}$, $\widetilde{O^+_{\tilde{\mathbf{E}}}}(F,I)$ has a group structure by composition.

remark 1. *Fixing a basis* $\alpha_0, \ldots, \alpha_l, a$, $\frac{\partial \bar{f}_0}{\partial \tau}(\tau)$ *is an automorphic factor* $(c\tau + d)^{-2}$. *If this automorphic factor is not degree* -2, *then the Proposition 3.6. does not hold.*

remark 2. *The Weyl group* $\tilde{W}_R$ *acts on the space* $\tilde{\mathbf{E}}$ *faithfully satisfying the above condition (3.4.1)(3.4.2), thus we have the natural inclusion map* $\iota : \tilde{W}_R \to \widetilde{O^+_{\tilde{\mathbf{E}}}}(F,I)$.

Under the above preparations, we define the central extension $\widetilde{O^+(F,I)}$ of $O^+(F,I)$. Let φ be the natural projection of $O^+(F,I) \to O^+(F,I)/\{\pm 1\}$.

Definition 3.6.

$$\widetilde{O^+}(F,I) := \{(x,y) \in \widetilde{O^+_{\tilde{\mathbf{E}}}}(F,I) \times O^+(F,I) | \psi(x) = \varphi(y).\}.$$

$\widetilde{O^+}(F,I)$ has a group structure in a natural way. We call the next two natural projections p_1 and p_2:

$$\begin{array}{rcl} p_1 : \widetilde{O^+}(F,I) & \to & \widetilde{O^+_{\tilde{\mathbf{E}}}}(F,I) \\ (x,y) & \mapsto & (x) \end{array} \tag{3.4.4}$$

$$\begin{array}{rcl} p_2 : \widetilde{O^+}(F,I) & \to & O^+(F,I) \\ (x,y) & \mapsto & (y) \end{array} \tag{3.4.5}$$

We define the action of $\widetilde{O^+}(F,I)$ on $\tilde{\mathbf{E}}$ throught p_1.

remark. *We have a natural embedding:*

$$\begin{array}{rcl} W_R & \to & \widetilde{O^+}(F,I) \\ g & \mapsto & (\iota(g), g|_F) \end{array} \tag{3.4.6}$$

Hereafter we regard $\tilde{W}_R$ as a subgroup of $\widetilde{O^+}(F,I)$ by the above homomorphism (3.4.6).

Proposition 3.7. *$\widetilde{O^+}(F,I)$ is a central extension of $O^+(F,I)$. We have the following diagram.*

$$\begin{tikzcd} & & 1 \arrow[d] & 1 \arrow[d] & \\ & & \{\pm 1\} \arrow[d] \arrow[r, equal] & \{\pm 1\} \arrow[d] & \\ 0 \arrow[r] & \mathbf{C} \arrow[r] \arrow[d, equal] & \widetilde{O^+}(F,I) \arrow[r, "p_2"] \arrow[d, "p_1"] & O^+(F,I) \arrow[r] \arrow[d, "\varphi"] & 1 \\ 0 \arrow[r] & \mathbf{C} \arrow[r] & \widetilde{O^+_{\tilde{\mathbf{E}}}}(F,I) \arrow[r, "\psi"] \arrow[d] & O^+(F,I)/\{\pm 1\} \arrow[r] \arrow[d] & 1 \\ & & 1 & 1 & \end{tikzcd} \tag{3.4.7}$$

Proof. We should only prove the exactness of the third row sequence. The other part of the proof of the diagram (3.4.7) is automatic.

We take one trivialization of the affine bundle $\tilde{\mathbf{E}}$.

$$\tilde{\mathbf{E}} \simeq \mathbf{E} \times \mathbf{C} \ni (x,t). \tag{3.4.8}$$

For all $f_0 \in O^+(F,I)/\{\pm 1\}$, we must study the existence and the ambiguity of the function $g(x,t)$ such that

$$\begin{array}{ccc} \mathbf{E}\times\mathbf{C} & \mapsto & \mathbf{E}\times\mathbf{C} \\ (x,t) & \mapsto & (f_0(x), g(x,t)) \end{array} \tag{3.4.9}$$

is an element of $\widetilde{O^+_{\tilde{\mathbf{E}}}}(F,I)$. The projection $p : F \to F/rad(I)$ induces the homomorphism $p : O^+(F,I) \to GL(F/rad(I))$. The image $p(O^+(F,I))$ is $O(F/rad(I), \bar{I})$, where $\bar{I}$ is a bilinear form on $F/rad(I)$ induced from I. By this fact, we can reduce the condition $f^*\tilde{I}^* = \frac{\partial \tilde{f}_0}{\partial \tau}(\tau)\tilde{I}^*$ (3.4.2) to the differential equation $dg(x,t) = \omega$.

The action $Aut^+(R)$ on $\mathbf{E}$ is a linear fractional transformation, therefore we find that ω is a closed 1-form. Since $\tilde{\mathbf{E}}$ is a simply connected space, we know that $g(x,t)$ always exists, and that the ambiguity of $g(x,t)$ is described by $\mathbf{C}$, the translation of the affine bundle $\tilde{\pi} : \tilde{\mathbf{E}} \to \mathbf{E}$.

$g(x,t)$ has a form of $t + g_1(x)$, therefore the condition (3.4.2) also implies that the map (3.4.9) becomes a bundle isomorphism of $\tilde{\pi} : \tilde{\mathbf{E}} \to \mathbf{E}$ as an affine bundle. *Q.E.D.*

remark. *$\widetilde{O^+}(F,I)$ acts on $\tilde{\mathbf{E}}$ transitively, therefore $\tilde{\mathbf{E}}$ has a structure of homogeneous space.*

Definition 3.8. *The central extension of $Aut^+(R)$ is defined as follows:*

$$\widetilde{Aut^+}(R) := (p_2)^{-1}(Aut^+(R)).$$

Theorem 3.9. *$\tilde{W}_R$ is a normal subgroup of $\widetilde{Aut^+}(R)$.*

Proof. For any $g \in \widetilde{O^+}(F,I)$, we shall prove

$$g\tilde{w}_\alpha g^{-1} = \tilde{w}_{\bar{g}\alpha}, \tag{3.4.10}$$

where $\alpha \in R$ and $\bar{g} = p_2(g) \in O^+(F, I)$. If the equation (3.4.10) holds, then we have the above theorem by considering only the cases $g \in \widetilde{Aut^+}(R)$.

If we send the both sides of the equation (3.4.10) by the homomorphism p_2, then the equation $p_2(g\tilde{w}_\alpha g^{-1}) = p_2(\tilde{w}_{\bar{g}\alpha})$ holds by the same calculation of the proof of proposition 3.2.. Hence we should only prove the equation

$$p_1(g\tilde{w}_\alpha g^{-1}) = p_1(\tilde{w}_{\bar{g}\alpha}). \tag{3.4.11}$$

In other words, we should only prove the equation (3.4.10) on $\tilde{\mathbf{E}}$.

First, we call the inverse image of the subgroup $O(F, rad(I)) \subset O^+(F, I)/\{\pm 1\}$ of the homomorphism (3.4.3) $\psi : \widetilde{O^+_{\tilde{\mathbf{E}}}(F, I)} \to O^+(F, I)/\{\pm 1\}$, $\widetilde{O(F, rad(I))}$.

This leads to the following disgram.

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathbf{C} & \to & \widetilde{O(F, rad(I))} & \to & O(F, rad(I)) & \to & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathbf{C} & \to & \widetilde{O^+_{\tilde{\mathbf{E}}}(F, I)} & \xrightarrow{\psi} & O^+(F, I)/\{\pm 1\} & \to & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & SL(rad(I))/\{\pm 1\} & = & SL(rad(I))/\{\pm 1\} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array} \tag{3.4.12}$$

By this diagram, we can reduce the proof to the two parts. 1. (3.4.11) holds for $g \in \widetilde{O(F, rad(I))}$. 2. (3.4.11) holds for some lifting of $SL(rad(I))/\{\pm 1\}$ to $\widetilde{O^+_{\tilde{\mathbf{E}}}(F, I)}$.

Lemma 3.10. *(3.4.11) holds for $g \in \widetilde{O(F, rad(I))}$.*

Proof. We prepare some notations. $F_{\mathbf{C}} := F \otimes_{\mathbf{R}} \mathbf{C}$. $\tilde{F}_{\mathbf{C}} := \tilde{F} \otimes_{\mathbf{R}} \mathbf{C}$. $\mathbf{R} \subset \mathbf{C}$ induces $F \subset F_{\mathbf{C}}$. $I_{\mathbf{C}} :=$ the $\mathbf{C}$ linear extention of I, which gives a bilinear form on $F_{\mathbf{C}}$. $\tilde{I}_{\mathbf{C}} :=$ the $\mathbf{C}$ linear extention of $\tilde{I}$, which gives a bilinear form on $\tilde{F}_{\mathbf{C}}$. We define $\widehat{O(F, rad(I))}$ as follows.

$$\widehat{O(F, rad(I))} := \{g \in O(\tilde{F}_{\mathbf{C}}, \tilde{I}_{\mathbf{C}}); g(F) \subset F,\ g|_F \in\ O(F, rad(I))\}. \tag{3.4.13}$$

Since $g \in \widehat{O(F,rad(I))}$ satisfies the conditions of $\widetilde{O(F,rad(I))}$, we have the homomorphism

$$(3.4.14) \qquad \widehat{O(F,rad(I))} \to \widetilde{O(F,rad(I))}.$$

For $g \in \widehat{O(F,rad(I))}$, (3.4.11) holds on $\tilde{F}$. Therefore (3.4.11) holds for the image of the homomorphism(3.4.13). We show that $\widetilde{O(F,rad(I))} = \widehat{O(F,rad(I))}$. Since $dim_{\mathbf{C}} rad(\tilde{I}) = 1$, we have the following commutative diagram.

$$(3.4.15) \qquad \begin{array}{ccccccccc} 0 & \to & \mathbf{C} & \to & \widehat{O(F,rad(I))} & \to & O(F,rad(I)) & \to & 1 \\ & & \| & & \downarrow & & \| & & \\ 0 & \to & \mathbf{C} & \to & \widetilde{O(F,rad(I))} & \to & O(F,rad(I)) & \to & 1 \end{array}$$

This implies that $\widehat{O(F,rad(I))} = \widetilde{O(F,rad(I))}$.

Q.E.D of lemma 3.10.

Lemma 3.11. *(3.4.11) holds for some lifting of $SL(rad(I))/\{\pm 1\}$ to $\widetilde{O}^{+}_{\tilde{\mathbf{E}}}(F,I)$.*

Proof. Fixing a basis (see (2.1)), we have one trivialization:

$$(3.4.16) \qquad \begin{array}{ccc} \tilde{\mathbf{E}} & \simeq & \mathbf{H} \times \mathbf{C}^l \times \mathbf{C} \\ (x) & \mapsto & (b(x), \alpha_1(x), \cdots, \alpha_l(x), \lambda(x)) \end{array}$$

We write the element of $\mathbf{H} \times \mathbf{C}^l \times \mathbf{C}$ by (τ, z, t). One lifting of $SL(rad(I)) \ni {}^t\begin{pmatrix} p & q \\ r & s \end{pmatrix}^{-1}$ is as follows:

$$(3.4.17) \qquad (\tau, z, t) \mapsto \left(\frac{p\tau + q}{r\tau + s}, \frac{z}{r\tau + s}, t + \frac{r < z, z >}{2(r\tau + s)}\right).$$

where $<,>$ is a $\mathbf{C}$-bilinear form induced from $\tilde{I}$. (3.4.11) can be proved for this lifting by the explicit calculation.

Q.E.D of lemma 3.11.

Q.E.D of Theorem 3.9.

Consequently, we obtain the following diagram.

$$
(3.4.18)\qquad
\begin{array}{ccccccccc}
 & & 0 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & \tilde{W}_R & \to & W_R & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathbf{C} & \to & \widetilde{Aut^+}(R) & \to & Aut^+(R) & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \mathbf{C}^* & \to & \widetilde{Aut^+}(R)/\tilde{W}_R & \to & Aut^+(R)/W_R & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}
$$

where $\mathbf{C}^*$ was normalized such that $\alpha \in \mathbf{C}^*$ acts on S_k as the multiplication of α^k. $\widetilde{Aut^+}(R)/\tilde{W}_R$ contains $\mathbf{C}^*$ as a center, thereby $\widetilde{Aut^+}(R)/\tilde{W}_R$ acts on S^W as a degree preserving transformation.

§4. The action of $\widetilde{Aut^+}(R)$ on the flat invariants.

In Saito [S-4], he introduced the non-degenerate $\mathbf{C}$ - metric J, and proved that the Levi-Civita connection ∇ with respect to J is integrable. He called the $\tilde{W}$ invariants associated to J, *the flat invariants.* In this section, we study the action of $\widetilde{Aut^+}(R)$ on the $\mathbf{C}$ - metric J, and in the last theorem, we shall write down the action of $\widetilde{Aut^+}(R)$ on the flat invariants explicitly.

In the rest of this paper, we assume that the codimension of the marked extended affine root system (R, G) equals one. (The notion of codimension was introduced in section 2.)

(4.1) Normalized lowest degree vector field and C-metric J.

In theorem 2.2, it was shown that there exist the algebraically free generators of the algebra S^W: $\Theta_0, \cdots, \Theta_l$. In the S^W- graded module Der_{S^W} , the lowest degree vector fields become a free $\Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}})-$ module of rank 1 (=codimension) generated by $\frac{\partial}{\partial \Theta_l}$. We normalize the ambiguity of multiplication of the $\Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}})-$ factor of Θ_l by the condition

$$(4.1.1)\qquad \frac{\partial^2}{\partial \Theta_l^2} \tilde{I}_W(d\Theta_l, d\Theta_l) = 0.$$

By (4.1.1), $\frac{\partial}{\partial\Theta_l}$ is determined uniquely up constant factor.

Hereafter we fix $\Theta_0, \cdots, \Theta_l$ such that Θ_l satisfies the condition (4.1.1). We define

$$T := \left\{ f \in S^W \middle| \frac{\partial}{\partial\Theta_l} f = 0 \right\} \tag{4.1.2}$$

$$Der_T := \text{the module of } \mathbf{C}- \text{ derivations of the algebra } T \tag{4.1.3}$$

$$\mathcal{G} := \left\{ \xi \in Der_{S^W} \middle| [\frac{\partial}{\partial\Theta_l}, \xi\,] = 0 \right\} \tag{4.1.4}$$

$$\mathcal{F} := \left\{ \omega \in \Omega^1_{S^W} \middle| L_{\frac{\partial}{\partial\Theta_l}} \omega = 0 \right\}, \tag{4.1.5}$$

where $L_{\frac{\partial}{\partial\Theta_l}}$ means the Lie derivative with respect to the vector field $\frac{\partial}{\partial\Theta_l}$. By one generators $\tau, \Theta_0, \cdots, \Theta_l$, we can represent $T, Der_T, \mathcal{G}, \mathcal{F}$ as follows.

$$T = \Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}})[\Theta_0, \cdots, \Theta_{l-1}] \tag{4.1.6}$$

$$Der_T = T\frac{\partial}{\partial\tau} \oplus \bigoplus_{i=0}^{l-1} T\frac{\partial}{\partial\Theta_i} \tag{4.1.7}$$

$$\mathcal{G} = T\frac{\partial}{\partial\tau} \oplus \bigoplus_{i=0}^{l} T\frac{\partial}{\partial\Theta_i} \tag{4.1.8}$$

$$\mathcal{F} = Td\tau \oplus \bigoplus_{i=0}^{l} Td\Theta_i \tag{4.1.9}$$

The pairing $Der_{S^W} \times \Omega^1_{S^W} \to S^W$ induces the complete pairing $\mathcal{G} \times \mathcal{F} \to T$, thereby $\mathcal{G}$ is a $T-$ dual module of $\mathcal{F}$.

We prepare one important lemma due to the Coxeter transformation theory. We recall that $\Theta_A := \tilde{J}(\tau, \Theta_0, \cdots, \Theta_l)$ is an anti $\tilde{W}_R$ invariants, whose degree equals $\frac{(l+2)(l_{max}+1)}{2}$ (we assumed that $cod(R, G) = 1$). Thus Θ_A^2 is an element of $\tilde{W}_R$ invariant function of degree $(l+2)(l_{max}+1) = (l+2)(deg\Theta_l)$. We expand Θ_A^2 by Θ_l:

$$\begin{aligned} \Theta_A^2 &= \phi(\tau, \Theta_0, \cdots, \Theta_l) \\ &= A_0\Theta_l^{l+2} + A_1\Theta_l^{l+2} + \cdots + A_{l+2} \end{aligned} \tag{4.1.10}$$

($A_i \in T$), then $A_0 \in \Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}})$ because of the degree condition.

Lemma 4.1 ([S-4](7.4)).

$$A_0(\tau) \neq 0 \quad {}^\forall \tau \in \mathbf{H}.$$

Sketch of proof. Let $c = \prod_{\alpha \in \Gamma_R} w_\alpha$ be a Coxeter transformation. Put $\mathbf{E}^c := \{x \in \mathbf{E} : c^*(x) = x\}$, where c^* is a dual of c. Put $\tilde{\mathbf{E}}^c := \tilde{\pi}^{-1}(\mathbf{E}^c), \tilde{c} = \prod_{\alpha \in \Gamma_R} \tilde{w}_\alpha$ (corresponding hyperbolic Coxeter transformation). Since $\tilde{\Theta}_i(= \lambda^{\tilde{m}_i}\Theta_i$ is an $\tilde{W}_R$-invariant function on $\tilde{\mathbf{E}}$, it is invariant under the action of a hyperbolic Coxeter transformation $\tilde{c}$. Using Lemma C and (2.9.5), we obtain,

$$\tilde{\Theta}_i(\tilde{c}\xi) = \tilde{\Theta}_i(\xi)(\xi \in \tilde{\mathbf{E}}^c)$$

therefore

$$\Theta_i(\xi) = exp(2\pi\sqrt{-1}\frac{deg\Theta_i}{l_{max}})\Theta_i(\xi).$$

If $i < l$, then $\Theta_i(\xi) = 0$. Thus we obtain,

$$\Theta_A^2(\xi) = A_0(\tau)\Theta_l^{l+2}(\xi) \ \ (\text{ where } \tau = \pi(\xi)).$$

Lemma B asserts that $\Theta_A^2(\xi) \neq 0$. Since $\pi|_{\mathbf{E}^c} : \mathbf{E}^c \to \mathbf{H}$ is surjective, we obtain $A_0(\tau) \neq 0 \quad {}^\forall \tau \in \mathbf{H}$. *Q.E.D.*

We define a T bilinear form,

$$(4.1.11) \qquad \begin{array}{cccc} J^* : \mathcal{F} \times \mathcal{F} & \to & T, \\ \omega_1 \times \omega_2 & \mapsto & \frac{\partial}{\partial \Theta_l}\tilde{I}(\omega_1, \omega_2). \end{array}$$

The value $\frac{\partial}{\partial \Theta_l}\tilde{I}(\omega_1, \omega_2)$ belongs to T by the condition (4.1.1). Then the next important fact was shown by the Coxeter transformation theory for the extended affine root system.

Proposition 4.2(Saito [S-4]). *The T bilinear form J^* is non-degenerate.*

Proof. By (4.1.1), all entries of $(l+2) \times (l+2)$ matrix $(\tilde{I}_W(d\tilde{\Theta}_i, d\tilde{\Theta}_j))_{i,j=-1,\cdots,l}$(where, we put $\tau = \Theta_{-1}$) are at most degree $= 1$ in $\tilde{\Theta}_l$. By the fact $det((\tilde{I}_W(d\tilde{\Theta}_i, d\tilde{\Theta}_j))_{i,j=-1,\cdots,l}) = \Theta_A^2 = A_0\tilde{\Theta}_l^{l+2} + \cdots + A_{l+2}$, we obtain $det(((\frac{\partial}{\partial \tilde{\Theta}_l}\tilde{I}_W(d\tilde{\Theta}_i, d\tilde{\Theta}_j))_{i,j=-1,\cdots,l}) = A_0(\tau)$. This does not vanish anywhere on $\mathbf{H}$ due to Lemma 4.1. *Q.E.D.*

Accordingly, we can also define the non-degenerate dual form

$$J : \mathcal{G} \times \mathcal{G} \to T. \tag{4.1.12}$$

(4.2) The action of $\widetilde{Aut^+}(R)$ on the C-metric J.

We shall study the transformation law for $\widetilde{Aut^+}(R)$ of J. We recall that $\widetilde{Aut^+}(R)$ acts on $\tilde{\mathbf{E}}$ throught p_1.

$$\begin{array}{cccc} p_1 : \widetilde{Aut^+}(R) & \to & \widetilde{Aut^+_{\tilde{\mathbf{E}}}}(R), \\ g & \mapsto & p_1(g). \end{array} \tag{4.2.1}$$

Proposition 4.3. *The transformation law of $\frac{\partial}{\partial \Theta_l}$ is as follows.*

$$(p_1(g^{-1}))_* \left(\frac{\partial}{\partial \Theta_l}\right) = \chi(g) \frac{\partial}{\partial \Theta_l}, \tag{4.2.2}$$

where χ is a group homomorphism:

$$\chi : \widetilde{Aut^+}(R) \to \mathbf{C}^*. \tag{4.2.3}$$

Proof. Since the action of $\widetilde{Aut^+}(R)$ on $Der_{\mathcal{S}W}$ is a degree preserving transformation, there exists a non-vanishing holomorphic function $f(\tau)$ on $\mathbf{H}$ satisfying $(p_1(g^{-1}))_*(\frac{\partial}{\partial \Theta_l}) = f(\tau)\frac{\partial}{\partial \Theta_l}$. We should only claim that $f(\tau) = const. \in \mathbf{C}$. We apply $p_1(g)$ on the both sides of equality (4.1.1), we have

$$\begin{aligned} 0 &= p_1(g)^* \left[\frac{\partial^2}{\partial {\Theta_l}^2} \tilde{I}_W(d\Theta_l, d\Theta_l)\right] \\ &= f^2(\tau) \left(\frac{\partial \overline{p_1(g)}}{\partial \tau}\right)^{-1} \frac{\partial^2}{\partial {\Theta_l}^2} \tilde{I}_W(d(f^{-1}(\tau)\Theta_l + h), d(f^{-1}(\tau)\Theta_l + h)), \end{aligned} \tag{4.2.4}$$

where $\overline{p_1(g)}$ is the isomorphism of $\mathbf{H}$ induced by $p_1(g)$ (see (3.4)), and h is an element of $\Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}})[\Theta_0, \cdots, \Theta_{l-1}]$. The element h does not affect on this term, $\tilde{I}_W(d\tau, d\tau) = 0$, the condition (4.1.1), and $\tilde{I}_W(d\tau, d\Theta_l) = \kappa^{-1}\Theta_l$ (2.8.11), thus we have

$$(R.H.S) = 2f^2(\tau) \left(\frac{\partial \overline{p_1(g)}}{\partial \tau}\right)^{-1} \frac{\partial f}{\partial \tau} \frac{1}{\kappa f^3(\tau)}. \tag{4.2.5}$$

Since $f(\tau), \left(\frac{\partial \overline{p_1(g)}}{\partial \tau}\right)^{-1}$ don't vanish, $f(\tau)$ therefore must be a constant. *Q.E.D.*

By Proposition 4.3, $\widetilde{Aut^+}(R)$ acts on T, $\mathcal{G}$ and $\mathcal{F}$.

Therefore the automorphism group $\widetilde{Aut^+}(R)$ acts also on J and J^* in a natural way.

Proposition 4.4. *The action of $\widetilde{Aut^+}(R)$ on J is as follows.*

$$p_1(g)^*J = \chi^{-1}(g)\left(\frac{\partial \overline{p_1(g)}}{\partial \tau}\right) J. \tag{4.2.6}$$

Proof. It's easy to see from Proposition 4.3., and the transformation property (3.4.2) of I. Q.E.D.

(4.3) The Levi Civita connection with respect to J.

We define the Levi Civita connection ∇ on $\mathcal{G}$ with respect to J.

Proposition 4.5. *(Saito [S-4]) There exists uniquely a torsion free, integrable, metric (w.r.t.J) connection ∇ on $\mathcal{G}$ as a T-module,*

$$\begin{array}{rccc} \nabla : & Der_T \times \mathcal{G} & \to & \mathcal{G}, \\ & (\delta_1, \delta_2) & \mapsto & (\nabla_{\delta_1}\delta_2). \end{array} \tag{4.3.1}$$

i.e.

0) The map $\nabla_\delta v$ is $T-$ linear in δ and satisfies the Leibniz rule:

$$\nabla_\delta(fv) = \delta(f)v + f\nabla_\delta v \text{ for } f \in T. \tag{4.3.2}$$

1) For $^\forall\delta, \xi \in Der_T$,

$$[\nabla_\delta, \nabla_\xi] = \nabla_{[\delta,\xi]}. \tag{4.3.3}$$

2) For $^\forall u, v \in \mathcal{G}$

$$\nabla_{\overline{u}} v - \nabla_{\overline{v}} u = [u, v]. \tag{4.3.4}$$

where $\overline{u}$ and $\overline{v}$ are the images of u and v in Der_T by the projection map:

$$\mathcal{G} \to \mathcal{G}/T\frac{\partial}{\partial \Theta_l} \simeq Der_T. \tag{4.3.5}$$

3) For $\delta \in Der_T$ and $u, v \in \mathcal{G}$,

$$\delta J(u, v) = J(\nabla_\delta u, v) + J(u, \nabla_\delta v). \tag{4.3.6}$$

Hereafter we fix one basis $a, b, \in rad(I) \cap Q(R)(a \in G \cap Q(R))$, thereby we also fix the coordinate function $\tau \in \Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}})$. We represent $\rho \circ \pi(g) \in \Gamma^+$ by a matrix

$$(\rho \circ \pi(g)b, \rho \circ \pi(g)a) = (b, a)^t \begin{pmatrix} p & q \\ r & s \end{pmatrix}^{-1}. \tag{4.3.7}$$

Proposition 4.6. *The action of $\widetilde{Aut^+}(R)$ on $\nabla_{\delta_1}\delta_2$ is as follows:*

$$p_1(g^{-1})_*(\nabla_{\delta_1}\delta_2) = \nabla_{\tilde{\delta}_1}\tilde{\delta}_2 - \frac{r}{r\tau + s}\left[(\tilde{\delta}_1\tau)\tilde{\delta}_2 + (\tilde{\delta}_2\tau)\tilde{\delta}_1 - J(\tilde{\delta}_1, \tilde{\delta}_2)\kappa^{-1}\frac{\partial}{\partial\Theta_l}\right], \tag{4.3.8}$$

where $\tilde{\delta}_i = p_1(g^{-1})_*\delta_i$ $(i = 1, 2)$ and $\delta_1 \in Der_T, \delta_2 \in \mathcal{G}$.

Proof. This follows from Proposition 4.4. and the formula

$$\begin{aligned} &2J(\nabla_{\delta_1}\delta_2, \delta_3) \\ &= \delta_1 J(\delta_2, \delta_3) + J(\delta_2, \ [\delta_3, \delta_1]) - \delta_3 J(\delta_2, \delta_1) - J([\delta_2, \delta_3], \delta_1) + \delta_2 J(\delta_3, \delta_1) + J(\delta_3, [\delta_1, \delta_2]), \end{aligned}$$

for $\delta_1 \in Der_T, \delta_2, \delta_3 \in \mathcal{G}$. and $J(\frac{\partial}{\partial\Theta_l}, \delta) = \kappa\delta\tau$ (see Saito [S-4, p52 assertion]) for $\delta \in \mathcal{G}$. *Q.E.D.*

(4.4) Modular property for the flat invariants.

We rewrite the degrees $0, \tilde{m}_0, \cdots, \tilde{m}_l$ of $\tau, \Theta_0, \cdots, \Theta_l$ as follows:

$$\begin{aligned} &0 = m_{0,1} =, \cdots, m_{0,n_0} < m_{1,1} = m_{1,2} =, \cdots, = m_{1,n_1} < m_{2,1} =, \cdots, = m_{2,n_2} <, \cdots, < \\ &m_{k,1} = m_{k,2} =, \cdots, = m_{k,n_k} < m_{k+1,1} =, \cdots, = m_{k+1,n_{k+1}}. \end{aligned} \tag{4.4.1}$$

such that $\tilde{m}_i = m_{p,q}$ when $i = q + \sum_{j=0}^{p-1} n_j$. By the assumption, codimension $= 1, n_{k+1} = 1$ and the duality of the exponents holds for $cod(R, G) = 1$ case. *i.e.*

$$n_i = n_{k+1-i} \ (0 \leq i \leq k+1) \tag{4.4.2}$$

Theorem 4.7 (flat invariants (Saito [S-4])). *In the module S^W, there exists uniquely a complex graded vector space V of rank $l+2$, whose weights are $0 = m_{0,1}, m_{1,1}, \ldots, m_{k+1,1}$ i.e.*

$$V = \bigoplus_{i=0}^{k+1} V_i \ \text{ where } dimV_i = n_i \ (0 \leq i \leq k+1) \tag{4.4.3}$$

such that

1. $V_0 = \mathbf{C}\tau$.
2. $S^W = \Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}}) \otimes_{\mathbf{C}[\tau]} S[V]$. *where* $S[V]$ *is a symmetric tensor algebra of* V.
3. $dV \subset \Omega^1_{S^W}$ *becomes a set of horizontal section of* $\mathcal{F}$ *with respect to the dual connection of* ∇.
4. J^* *defines a non-degenerate* $\mathbf{C}$*-bilinear form on* V *using the inclusion map:* $V \hookrightarrow dV \subset \Omega^1_{S^W}$

$$J^* : V \times V \to \mathbf{C}, \tag{4.4.4}$$

in particular J^* *defines a complete pairing of* V_i *and* V_{k+1-i} $(0 \le i \le k+1)$.

$$J^* : V_i \times V_{k+1-i} \to \mathbf{C}. \tag{4.4.5}$$

We call the elements of V *the flat invariants.*

Theorem 4.8 (modular property for the flat invariants). *The action of* $\widetilde{Aut^+}(R)$ *on the flat invariants is as follows: for* $(1 \le i \le k)$

$$p_1(g)^*\tau = \frac{p\tau + q}{r\tau + s}, \tag{4.4.6}$$

$$p_1(g)^* v_i = \frac{1}{r\tau + s} A_i(g^{-1}) v_i \quad \text{for all } v_i \in V_i, \tag{4.4.7}$$

$$p_1(g)^*\hat{\Theta}_l = \chi(g^{-1})\left[\hat{\Theta}_l + \frac{r}{2\kappa(r\tau + s)} \sum_{i,j=0}^{l-1} J\left(\frac{\partial}{\partial\hat{\Theta}_i}, \frac{\partial}{\partial\hat{\Theta}_j}\right)\hat{\Theta}_i\hat{\Theta}_j\right], \tag{4.4.8}$$

Also A_i *has a duality with respect to* J^*:

$$J^*(A_i(g^{-1})v_i, A_{k+1-i}(g^{-1})v_{k+1-i}) = \chi(g) J^*(v_i, v_{k+1-i}) \text{ for all } v_i \in V_i. \tag{4.4.9}$$

where

1. $A_i : \widetilde{Aut^+}(R) \to GL(V_i)$ *is a group homomorphism.*
2. $\{\hat{\Theta}_0, \cdots, \hat{\Theta}_l\}$ *is a union of the basis of* $\bigoplus_{i=1}^k V_i$, *and* $\hat{\Theta}_l \in V_k$.
3. κ *is a non-zero constant defined in (2.8.11).*

4. $J(\frac{\partial}{\partial\hat{\Theta}_i}, \frac{\partial}{\partial\hat{\Theta}_j}) \in \mathbf{C}$

Proof. By proposition 4.6., we can calculate the above results. The duality (4.4.9) is a direct consequence of the proposition 4.4. and the equation (4.4.6). *Q.E.D.*

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